Abstract Algebra Problemset

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1 Section 1. Categories and functors

1.1 Problem 1

Let f be a morphism in a category C. Prove the following:

1. **If f is an isomorphism then f is a monomorphism and an epimorphism.**
2. **The inclusion of $\mathbb{Z}$ in $\mathbb{Q}$ is a monomorphism and an epimorphism in the category of rings but not an isomorphism**

Part (1). $f \in Hom_C(A, B)$ is a isomorphism iff $\exists g \in Hom_C(B, A)$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$. Hence,

$$f \circ h = f \circ h' \implies g \circ f \circ h = g \circ f \circ h' \implies h = h'$$

so f is a monomorphism. Equivalently over the left side, one can see f is an epimorphism.

Part (2). $i : \mathbb{Z} \mapsto \mathbb{Q}$ is a monomorphism as it acts as the identity to the elements of $\mathbb{Z}$. If there are two ring morphism $f, g : A \mapsto Z$ for which $i(f(a)) = i(g(a)) \forall a \in A$, then $f(a) = g(a) \forall a \in A$, so i is mono.
It's also an epimorphism as, for any two ring morphisms $h, h' : \mathbb{Q} \mapsto C$

$$h \circ i = h' \circ i \implies h(n) = h'(n) \text{ with } n \in \mathbb{Z}$$
$$\implies h(q) = h(a/b) = h(a)/h(b) = h'(a)/h'(b) = h'(a/b) = h'(q) \text{ for all } q \in \mathbb{Q}$$

Last, it is not a isomorphism of rings as if it were, there would exist an inverse $g : \mathbb{Q} \mapsto \mathbb{Z}$ morphism of rings. This is not possible as $1 = g(2 \cdot 1/2) = 2 \cdot g(1/2)$, which would imply $2 \mid 1$.

1.2 Problem 2

Show that in the category of finite dimensional vector spaces over a field $\mathbb{K}$ we have a natural equivalence of functors between the identity Id and the bidual $(-)^{**}$.

Restating the equivalence of functors in the terms of $\mathbb{K}$-vector spaces, we need to prove that

1. There is an isomorphism, $\tau_V : V \mapsto V^{**}$ for all V a finite dimensional vector spaces.

   (proof) We define a linear map $\tau_V : V \mapsto V^{**}$ that, given a vector u, returns $\tau_V(u) := \phi_u$, a linear form $\phi_u : V^* \mapsto \mathbb{R}$ such that $\phi_u(w) = w(u)$. It is injective as if $\phi_u(w) = w(u) = 0$ for all forms $w \implies u = 0$. From the dual basis theorem, we know the $\dim V = \dim V^* = \dim V^{**}$, so injective $\implies$ isomorphism.

2. The bidual map of $f : V \mapsto W$, which is $f^{**} : V^{**} \mapsto W^{**}$, obeys the following commutative property $f^{**} \equiv \tau_W \circ f \circ \tau_V^{-1}$

   (proof) Equivalently, we will see $f^{**} \circ \tau_V = \tau_W \circ f \iff \forall v \in V \;\; f^{**}(\phi_v) = \phi_{f(v)} \iff \forall w \in W^* \;\; f^{**}(\phi_v)(w) = w(f(v))$, which exactly the definition of the bidual map.

1.3 Problem 3

Show that two categories $\mathcal{B}$ and $\mathcal{C}$ are naturally equivalents if and only if there exists a fully faithful and essentially surjective covariant functor $F : \mathcal{B} \to \mathcal{C}$.

$\boxed{\implies}$ A, B are natural equivalents $\implies \exists F : A \mapsto B$ fully faithful and dense.
By the natural equivalence of categories, there exist two functors $F : A \mapsto B$ and $G : B \mapsto A$ such that $F \circ G \simeq Id_B$ and $G \circ F \simeq Id_A$. This means that for all objects $S, T \in A$ and $f \in Hom_A(S,T)$ there are isomorphisms τ_S, τ_T such that $\tau_T \circ (G \circ F)(f) \circ \tau_S^{-1} = f$. And for all objects $C, D \in B$ and $g \in Hom_B(C,D)$ there are isomorphisms π_C, π_D such that $\pi_D \circ (F \circ G)(g) \circ \pi_S^{-1} = g$. We will see that F is fully faithful and dense.

1. *Faithful.* Suppose $F(f) = F(f') \implies G \circ F(f) = G \circ F(f') \implies \tau_T \circ f \circ \tau_S^{-1} = \tau_T \circ f' \circ \tau_S^{-1} \implies f = f'$. Note that equivalently G is faithfull, which we will use in the proof of F full.

2. *Dense.* For all $C \in ob(B)$, take $S = G(C)$. Then $F(S) = F \circ G(C)$ which is isomorphic to C by π_C.

3. *Full.* Given $S, T \in \text{obj}(A)$. For all $g \in Hom_B(F(S), F(T))$ there is an $f \in Hom_A(S,T)$ such that $F(f) = g$. Let $f = \tau_T^{-1} \circ G(g) \circ \tau_S$. Then

$$F(f) = F(\tau_T^{-1}) \circ (F \circ G)(g) \circ F(\tau_S)$$

   Now we will use that G is faithful and prove $(G \circ F)(f) = G(g)$ which will imply $F(f) = g$.

$$(G \circ F)(f) = (G \circ F)(\tau_T^{-1}) \circ (G \circ F \circ G)(g) \circ (G \circ F)(\tau_S)$$

   By the naturallity of the choice of morphisms τ, $(G \circ F)(\tau_T) = \tau_{G \circ F(T)}$ and because functors preserve inverses,

$$(G \circ F)(f) = \tau_{G \circ F(T)}^{-1} \circ (G \circ F)(G(f)) \circ \tau_{G \circ F(S)} = G(f)$$

   which concludes the proof.

$\boxed{\impliedby}$ $\exists F : A \mapsto B$ fully faithful and dense $\implies$ A, B are natural equivalents
We construct the inverse functor $G : B \mapsto A$ such that $\forall C \in ob(B)$, $G(C) = S$, the object such that $F(S) \simeq C$ which is ensured by F being *Dense*. We note the isomorphism between $C \simeq F(G(C))$, π_C. Then, for all $g : C \mapsto D$, we take $G(g) = F^{-1}(\pi_D \circ g \circ \pi_C^{-1})$, which exists and is unique as F is a bijection between $\text{Hom}_A(S,T) \mapsto \text{Hom}_B(F(S), F(T))$ (F is *Fully Faithful*). This defines G completely.
To see that G is a functor, we need

1. Identity goes to identity. $G(\text{id}_C) = F^{-1}(\pi_C \circ \text{id}_C \circ \pi_C^{-1}) = F^{-1}(\text{id}) = \text{id}$. In the last equality we have used that F is a functor so it sends identities to identities (and the inverse on the morphisms is unique).

2. Composition is mantained. $G(f \circ g) = F^{-1}(\pi_S \circ f \circ g \circ \pi_R^{-1}) = F^{-1}(\pi_S \circ f \circ \pi_T^{-1} \circ \pi_T \circ g \circ \pi_R^{-1}) = F^{-1}(\pi_S \circ f \circ \pi_T^{-1}) \circ F^{-1}(\pi_T \circ g \circ \pi_R^{-1}) = G(f) \circ G(g)$.

The equivalence of functors between $G \circ F \simeq Id_B$ and $F \circ B \simeq Id_A$ is direct by construction. The isomorphisms τ_S in category A and π_C in category B are the ones ensured by F dense. The commutative property $(G \circ F)(f) = \tau_T^{-1} \circ f \circ \tau_S$ is given by construction.

1.4 Problem 4

Pullbacks in the category of abelian groups: Let A and B be abelian groups together with homomorphisms $f : A \to S$ and $g : B \to S$. Prove that $A_S B = \{(a,b) \in AB | f(a) = g(b)\}$.

Translating the universal property to the category of Abelian groups, we need to check

1. $A \times_S B$ is an Abelian group. That π_A and π_B, the projection onto the first and second coordinate, are group morphisms.

   (proof) The restriction is closed under product and inverse, as f, g are group homomorphisms. Product $(a,b), (c,d) \in A \times_S B$, $f(ac) = f(a)f(c) = g(b)g(d) = g(bd)$. Inverse $f(a^{-1}) = f(a)^{-1} = g(b)^{-1} = g(b^{-1})$. Hence $A \times_S B \subset A \times B$, which is abelian. π_A and π_B are clearly morphisms by the universal property of the product and because $A \times_S B$ is a subgroup of the product.

2. $\exists s : A \times_S B \mapsto S$ such that $s = f \circ \pi_A = g \circ \pi_B$

   (proof) $s((a,b)) := f(a) = g(b)$, which is well defined, morphism and satisfies the condition as $(a,b) \in A \times_S B$.

3. For all Abelian group C and morphisms $h_A : C \mapsto A, h_B : C \mapsto B$ such that $h_A \circ f = h_b \circ g$, there exists a unique $\phi : C \mapsto A \times_S B$ such that $\pi_A \circ \phi = h_A$ and $\pi_B \circ \phi = h_B$.

   (proof) Define $\phi : C \mapsto A \times_S B$ as $\phi(c) := (h_A(c), h_B(c))$. It is well defined as $f \circ h_A = g \circ h_B$, so it actually maps to $A \times_S B \subset A \times B$. It obviously satisfies the commutative property by construction. It is unique as $\pi_A \circ \phi = h_A$ implies that the first component of ϕ must be h_A and idem for the second and B. We are implicitly using the universal property of products, any group homomorphism to the product can be split uniquely into component homomorphisms and viceversa.

1.5 Problem 5

Pushouts in the category of abelian groups: Let A and B be abelian groups together with homomorphisms $f : S \to A$ and $g : S \to B$. Prove that:

$$A \sqcup_S B = \frac{A \oplus B}{W}$$

where W is the subgroup generated by $(f(s), -g(s))$ with $s \in S$.

Translating the universal property to the category of Abelian groups, we need to check

1. $A \sqcup_S B$ is an Abelian group. Let $\lambda_A : A \mapsto A \sqcup_S B$ such that $\lambda(a) = \overline{(a, 0)}$ and idem for λ_B. They are the natural quotient maps.

   (proof) Direct sum and quotient by a subgroup W conserve commutativity.

2. $\exists z : S \mapsto A \sqcup_S B$ such that $z(s) = \lambda_A \circ f = \lambda_B \circ g$

   (proof) $\lambda_A \circ f(s) = \overline{(f(s), 0)} = \overline{(f(s), 0)} + \overline{(-f(s), g(s))} = \overline{(0, g(s))} = \lambda_B \circ g(s)$

3. For all Abelian group C and morphisms $h_A : A \mapsto C, h_B : B \mapsto C$ such that $h_A \circ f = h_B \circ g$ there exists a unique $\phi : A \sqcup_S B \mapsto C$ such that $\phi \circ \lambda_A = h_A$ and $\phi \circ \lambda_B = h_B$.

   (proof) Define $\phi : A \sqcup_S B \mapsto C$ as $\phi(\overline{(a, b)}) = h_A(a) + h_B(b)$. It is well defined as if $(a, b) - (c, d) \in W \implies (a - c, b - d) = \sum(f(s_i), -g(s_i)) = (f(s), -g(s))$. Hence, $h_A(a - c) + h_B(b - d) = h_A(f(s)) - h_B(g(s)) = 0$. Hence, its a morphism. Suppose it is not unique $\exists \phi, \phi'$ with the universal property. Then $(\phi - \phi') \circ \lambda_A = 0 = (\phi - \phi') \circ \lambda_B \implies (\phi - \phi')(\overline{(a, b)}) = 0 \implies \phi \equiv \phi'$.

1.6 Problem 6

Inverse limits in the category of sets / groups / abelian groups / modules: Let $(\{A_i\}, \{f_j, i\})$ be an inverse system over a preordered set I. Prove that

$$\varprojlim A_i = \{(a_i) \in \prod A_i | f_{j,i}(a_j) = a_i, i \leq j\}$$

Again, translating the universal property to the categories that matter to this problem, we need to check.

1. $\varprojlim A_i$ a is set / group / abelian groups / modules. That the $\pi_j : \varprojlim A_i \mapsto A_j$, the natural projections from the product, are morphisms and $\pi_i = f_{j,i} \circ \pi_j \forall i, j \; i \leq j$.

   (proof) All these categories accept products and our inverse limit is a product with some additional constraints. Because the additional constraints are morphisms, they are closed with respect operations, so the final product will be a set / group / abelian group / module. By the universal property of the product and because the extra restrictions are equalities of morphisms, the π_i are morphisms. Lastly, $\pi_i = \pi_j \circ f_{i,j} \forall i, j \; i \leq j$ is true as the tuples in the limit (a_i) satisfy $f_{j,i}(a_j) = a_i$.

2. For all object B and morphisms $f_j : B \mapsto A_j$ that satisfy the inverse ordering $f_j = f_{j,i} \circ f_i$ for $i \leq j$, there is a unique $\phi : B \mapsto \varprojlim A_i$ such that $\pi_i \circ \phi = f_i$.

   (proof) Take $\phi(b) := (f_i(b))_i$. It's well defined (it maps onto the inverse limit) as the family f_i follow the inverse ordering for all points. It follows the commutative property by construction and it is a morphism as it is formed by components that are morphism. It is unique as every component function must be $\phi_i = \pi_i \circ \phi = f_i$, by the commutative property.

1.7 Problem 7

Direct limits in the category of sets / groups / abelian groups / modules / rings with unit: Let $(\{Ai\}, \{f_{i,j}\})$ be an inverse system over a directed set I. Prove that

$$\varinjlim A_i = \sqcup A_i / \sim$$

where $a_i \sim a_j$ iff $f_{i,l}(a_i) = f_{j,l}(a_j)$ for $i, j \leq l$

Again, translating the universal property to the categories that matter to this problem, we need to check

1. $\varinjlim A_i$ is a set / group / abelian group / module. That $\lambda_j : A_j \mapsto \varinjlim A_i$ such that $\lambda_j(a) = \overline{a}$ are morphisms and $\lambda_i = \lambda_j \circ f_{i,j} \forall i, j \; i \leq j$.

   (proof) Given $a_i \in A_i$, $\lambda_i(a_i) = \overline{a_i}$ and $\lambda_j \circ f_{i,j}(a_i) = \overline{f_{i,j}(a_i)}$. They are equivalent under $\sim$ because, by the definition of direct system, for all $k \geq i, j$ $f_{i,k}(a_i) = f_{j,k}(f_{i,j}(a_i))$. The object is still in the category as all of the categories accept disjoint unions (this would not happen, for example, in the case of fields) and we are doing the quotient by an equivalence relation that respects the internal operations of the object (because it is an equality of morphisms).

2. For all object B and morphisms $f_j : A_j \mapsto B$ that satisfy the direct order $f_i = f_j \circ f_{ij}$ if $i \leq j$, there is unique morphism $\phi : \varinjlim A_i \mapsto B$ such that $\phi \circ \lambda_i = f_i$.

   (proof) Construct $\phi : \varinjlim A_i \mapsto B$ such that $\phi(\overline{a_k}) = f_k(a_k)$. It is well defined over the quotient as if

$$\overline{a_k} \equiv \overline{b_r} \implies f_{k,l}(a_k) = f_{r,l}(b_r) \; \exists l \;\; k, r \leq l$$

   ,

   then $\phi(\overline{a_k}) = f_k(a_k) \underset{\text{by comm.}}{=} f_l \circ f_{k,l}(a_k) \underset{\text{by equiv.}}{=} f_l \circ f_{r,l}(b_r) = f_r(b_r) = \phi(\overline{b_r})$. Clearly $\phi \circ \lambda_i(a_i) \underset{\text{by def.}}{=} f_i(a_i)$

1.8 Problem 8

Show that in an abelian category we have:

1. **f is a monomorphism iff $ker(f) = 0$.**
2. **f is an epimorphism iff $Coker(f) = 0$.**
3. **A monomorphism is the kernel of its cokernel.**
4. **An epimorphism is the cokernel of its kernel.**
5. **Every morphism can be expressed as the composition of an epimorphism and a monomorphism.**
6. **f is an isomorphism iff f is an epimorphism and a monomorphism.**

Part (1). f monomorphism $\iff$ Ker$f = 0$.
$\boxed{\implies}$ As f is a monomorphism and $f \circ i = 0 = f \circ 0 \implies i = 0$. To see that this implies Ker$f = 0$, we need to see that the tuple $(0, 0)$ is a valid kernel and, by uniqueness of the categorical definition, it will be the kernel. Effectively

1. $f \circ i = f \circ 0 = 0$
2. For all $g \in Hom(S, A)$ such that $f \circ g = f \circ 0 = 0 \underset{f\text{mono}}{\implies} g = 0$ there exists a unique $\overline{g} = 0$ such that $i \circ \overline{g} = g \iff 0 \circ 0 = 0$

$\boxed{\impliedby}$ If $f \circ h = f \circ h' \implies f \circ (h - h') = 0$ which, by the universal property of the kernel, implies that $\exists! g$ such that $i \circ g = h - h'$. As Ker$f = 0 \implies i = 0$, hence $i \circ g = 0 = h - h' \implies h = h'$.

Part (2). Idem to Part 1, one can see that $(0, 0)$ is a Coker of f

Part (3). I need to see that, under the hypothesis of f mono, (A, f) is a valid kernel of the cokernel.

1. $\pi \circ f = 0$ which is true by the definition of Coker.

2. $\forall g \in Hom(C, B)$ such that $\pi \circ g = 0$, there exists a unique $\overline{g}$ such that $f \circ \overline{g} = g$. Let $\overline{g} = \tau^{-1} \circ \overline{f}^{-1} \circ j \circ g$. Then $f \circ \overline{g} = g$ and, by f mono, $\overline{g}$ is unique, as we wanted.

Part (4). Idem to Part 3, one can see that (B, f) is a valid Cokernel of i.

Part (5). Take the composition $f = (j \circ \overline{f}) \circ \tau$ or $f = j \circ (\overline{f} \circ \tau)$. This works as j is mono, τ is epi and $\overline{f}$ is iso.

Part (6). One implication is always true, as proven in Exercise 1. For the other, using (3), we can see that f is the kernel of its cokernel, so it is the morphism v from the Ker CoKer f = Im f to B. Using (4) we can see that f is the morphism w from A to CoKer Ker f = CoIm f. As the category is Abelian, $\overline{f}$ is an isomorphism, so Im f is isomorphic to CoIm f so the starting object of f is isomorphic to the ending space and f is an isomorphism.

2 Section 2. Modules

2.1 Problem 1

Nakayama's lemma. Let M be a finitely generated A-module and I an ideal of A contained in the Jacobson radical $= \cap \mathcal{M}$, $\mathcal{M}$ maximal ideal. Prove: $IM = M \implies M = 0$.

Supose $M \neq 0$. As it is finitely generated and non zero, let $M = \langle m_1, \cdots, m_r \rangle$ a minimal set of A-generators of M with $r \geq 1$. Using the hypothesis, $\forall m \in M = IM \implies \exists s \in I, m' \in M$ such that $m = sm'$ and, because M is finitely generated, $m' = \sum^r a_i m_i$, which implies that $m = \sum^r (sa_i) m_i = \sum^r s_i m_i$, with $s_i \in I$. We have proven that with our hypothesis, the set of minimal A-generators of M are also a set of I-generators of M. In particular, $m_1 = \sum^r s_i m_i \implies m_1(1 - s_1) = \sum_2^r s_i m_i$. If we show $1 - s_1$ is invertible, m_1 would be expressed as a A-linear combination of $m_2, \cdots m_r$, which would contradict minimality.
Consider the ideal $(1 - s_1)$. If it's included in a maximal ideal F, as $s_1 \in R = \cap F_i \implies s_1 \in F \implies 1 = (1 - s_1) + s_1 \in F \implies F = A$, which is contradictory with proper maximality of F. Hence, $(1 - s_1) = A$, so there is a $t \in A$ such that $t(1 - s_1) = 1$, $1 - s_1$ is invertible finishing the proof.

2.2 Problem 2

Under the previous hypothesis, prove:

1. **$A/I \otimes_A M = 0 \implies M = 0$**
2. **If $N \subset M$ is a submodule, $M = IM + N \implies M = N$.**
3. **If $f : N \to M$ is a homomorphism, $\overline{f} : N/IN \to M/IM$ surjective $\implies f$ surjective.**

Part (1). From problem 11, $A/I \otimes_A M \simeq M/IM = 0 \implies M = IM$. From Problem 1, this implies $M = 0$.

Part (2). $M = IM + N$ implies, looking $\mod N$ that $M/N = (IM)/N$. Formally, any $m \in M$ can be expressed as $m = im' + n$ so for any class $\overline{m} = \overline{im' + n} = \overline{im'} = i\overline{m'}$.
Hence $M/N = I(M/N)$. By Problem 1, this implies $M/N = 0 \implies M = N$.

Part (3). The function is well defined as $f(IN) \in IM$, $f(in) = if(n)$. $\overline{f}$ surjective $\implies \overline{f}(N/IN) = M/IM$. Now, let's see $\overline{f}(N/IN) = f(N)/IM$.
$\boxed{\subseteq}$. For all $\overline{x} \in \overline{f}(N/IN)$ $\exists \overline{y} \in N/IN$ such that $\overline{f}(\overline{y}) = \overline{f(y)} = \overline{x}$. Hence $\overline{x} \in f(N)/IN$.
$\boxed{\supseteq}$. For all $\overline{x} \in f(N)/IN$ exist $i \in I, n, n' \in N$ such that $f(n) = x + in'$. Hence there is a $\overline{n} \in N/IN$ such that $\overline{f}(\overline{n}) = \overline{f(n)} = \overline{x + in'} = \overline{x} \implies \overline{x} \in \overline{f}(N/IN)$.
Lastly, $f(N)/IM = M/IM \implies f(N) - M = IM \implies M = IM + f(N) \implies M = f(N) \implies f$ surjective.

2.3 Problem 3

Let (A, m) be a local ring and M be a finitely generated A-module, $x_1, ..., x_n$ elements of M. Using Nakayama's lemma prove that:

1. **$x_1, ..., x_n$ generate M over $A \iff \overline{x}_1, ..., \overline{x}_n$ generate M/m over A/m.**

2. **$x_1, ..., xn$ is a minimal system of generators of $M \iff \overline{x}_1, ..., \overline{x}_n$ is a basis of the A/m-vector space M/mM.**

3. **All minimal systems of generators of M have the same number of elements, equal to the dimension of the A/m-vector space M/mM.**

4. **$x_1, ..., x_n$ are part of a minimal system of $M \iff x_1, ..., x_n$ are linearly independent in M/mM.**

Part (1). $\boxed{\implies}$ Given a $\overline{x} \in M/mM$, there are some $a_1, a_2, \cdots, a_n$ such that $x = \sum a_i x_i \implies \overline{x} = \sum \overline{a_i x_i} = \sum a_i \cdot \overline{x_i} = \sum \tilde{a}_i \cdot \overline{x_i}$, so $\overline{x_i}$ generate M/mM over A/m.

$\boxed{\impliedby}$ $M/mM = \langle x_1, \cdots, x_n \rangle_{A/m}$ so all $\overline{x} = \sum \tilde{a}_i \overline{x_i} \implies x = \sum a_i x_i + rx'$ with $r \in m$ and $x' \in M \implies M = \langle x_1, \cdots, x_n \rangle + mM \underset{\text{by Nakayama's Lemma}}{\implies} M = \langle x_1, \cdots, x_n \rangle$.

Part (2). $\boxed{\implies}$ Suppose not, then WLOG $\overline{x_1}, \cdots \overline{x_{n-1}}$ would generate M/mM so by Part (1), $x_1, \ldots x_{n-1}$ would be generators, contradicting minimallity.

$\boxed{\impliedby}$ Suppose not, WLOG $x_1, \cdots x_{n-1}$ are generators which, by Part (1), imply that $\overline{x_1}, \cdots, \overline{x_{n-1}}$ are a base, which contradicts $\overline{x_1}, \cdots, \overline{x_n}$ being a base, as $\overline{x_n}$ could be expressed as a linear combination of the smaller terms, contradicting linear independence.

Part (3). They are in bijection with bases of the vector space M/mM over the field A/m, in which (if it is finite) all bases have the same size, the dimension.

Part (4). $\boxed{\implies}$ Complete the minimal system, then by Part (2) you have a base and any subset of a base is linearly independent.

$\boxed{\impliedby}$ Complete to a base, then by Part (2) you have a minimal set of generators that contain the x_i.

2.4 Problem 4

Let A be a non-local ring. Prove that the A-module A has two minimal systems of generators with a different number of generators.

On one hand, we know that 1 is a minimal generator of A of cardinality 1. We will find another set of generators with a larger cardinality. As A is non-local (and non-degenerate), there exists at least two distinct maximal ideals I, J. We can take a (possibly infinite) minimal system of generators of $I = \langle i_1, \cdots \rangle$ and an element $j \in J, j \notin I$. Then, $\langle j, i_1 \cdots \rangle$ must generate A, (as I is maximal). This set is minimal as taking j out wouldn't generate the elements of I^c and taking i_r would contradict minmality of the $\langle i_j \rangle = I$. The cardinallity of this set is clearly > 1, so it is distinct to 1.

2.5 Problem 5

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Prove that if M' and M'' are finitely generated, then M is finitely generated.

Let $f : M' \mapsto M$ be injective, $g : M \mapsto M''$ surjective and $\text{Im} f = \text{Ker} g$.

Take $m \in M$. As M'' is finitely generated, $g(m) = \sum^n a_i m_i''$. Because g is surjective, $\exists m_i^1 \in M$ such that $g(m_i^1) = m_i''$ so the previous identity can be written as

$$g(m) = \sum^n a_i g(m_i^1) \implies g(m - \sum^n a_i m_i^1) = 0 \implies m - \sum^n a_i m_i^1 \in \text{Ker}g = \text{Im}f$$
$$\exists m' \in M' \text{ such that } f(m') = m - \sum a_i m_i^1.$$

As M' is finitely generated, $m' = \sum^{n'} b_j m_j' \implies f(m') = \sum^{n'} b_j f(m_j')$. Let $f(m_j') = m_j^2$. Reorganizing the equation above,

$$m = \sum^n a_i m_i^1 + \sum^{n'} b_j m_j^2 \tag{1}$$

which implies that the set $\langle m_1^1, \cdots m_n^1, m_1^2 \cdots m_{n'}^2 \rangle$ generates M, so it is finitely generated.

2.6 Problem 6

Prove that $\mathbb{Z}[\sqrt{d}]$ is a Noetherian ring.

We will use that Noetherian property is inferred from extremes to the center (and viceversa) in a short exact sequence. We build an exact sequence sandwitching $\mathbb{Z}[\sqrt{d}]$ between two $\mathbb{Z}$, which we know are Noetherian.
Let $f : \mathbb{Z} \mapsto \mathbb{Z}[\sqrt{d}]$ $f(n) = n$ and $g : \mathbb{Z}[\sqrt{d}] \mapsto \mathbb{Z}$ $g(a+b\sqrt{d}) = b$. For the definition of g we implicitly are using that d is a non-square and every element of $\mathbb{Z}[\sqrt{d}]$ can be uniquely expressed as $a + b\sqrt{d}$. Also note that if d is a square, $\mathbb{Z}[\sqrt{d}] = \mathbb{Z}$ which is Noetherian. Clearly, f is injective and g is exhaustive. $\text{Im}f = \{a + b\sqrt{d} | b = 0\} = \text{Ker}g$, so the short sequence is exact, which completes the proof.
It can be seen in an easier way by realizing $\mathbb{Z}[\sqrt{d}] \simeq \mathbb{Z}[T]/(T^2 - d)$ and, by Hilbert's base, $\mathbb{Z}[T]$ is Noetherian and the quotient of a Noetherian module is Noetherian. Noetherian is conserved by quotients because the short sequence on inclusion-quotient $0 \mapsto S \mapsto G \mapsto G/S \mapsto 0$ is exact.

2.7 Problem 7

Prove that the ring $\mathbb{Z}[2T, 2T^2, 2T^3, ...] \subset \mathbb{Z}[T]$ is not Noetherian.

We take the following family of ideals $A_i = (2T^i)$. Clearly $A_i \subseteq A_{i+1}$ but we will show it does not stabilize. Suppose it did, then for some n, $(2T^n) = (2T^{n+1}) \implies 2T^{n+1} \in (2T^n) \implies 2T^{n+1} = p(T)2T^n$ for some $p \in \mathbb{Z}[2T, 2T^2, \cdots]$. For the equality to hold, the degree of p must be exactly 1, as $\mathbb{Z}$ is a domain. Hence $p(T) = A \cdot 2T$ for some $A \neq 0$. We have that $A \cdot 4T^{n+1} = 2T^{n+1}$ which is a contradiction as the RHS is not divisible by 4.

2.8 Problem 8

Let M be an A-module and let N_1, N_2 be submodules of M. Prove that if M/N_1 and M/N_2 are Noetherian (Artinian) then $M/(N_1 \cap N_2)$ is Noetherian (Artinian) as well.

First, we construct the following short exact sequence

$$0 \mapsto N_1/(N_1 \cap N_2) \overset{f}{\mapsto} M/(N_1 \cap N_2) \overset{g}{\mapsto} M/N_1 \mapsto 0$$

where $f : N_1/(N_1 \cap N_2) \mapsto M/(N_1 \cap N_2)$ is the natural inclusion and $g : M/(N_1 \cap N_2) \mapsto M/N_1$ is pass to quotient. g is well defined as if $m - m' \in N_1 \cap N_2 \implies m - m' \in N_1$. f is mono and g is epi trivially. It remains to see that $\text{Ker} g = \text{Im} f$,

$$\overline{n} \in \text{Ker} g \iff g(\overline{n}) = \tilde{n} = \tilde{0} \iff n \in N_1 \iff \overline{n} \in N_1/(N_1 \cap N_2) \iff \overline{n} \in \text{Im} f$$

It remains to be seen that $N_1/(N_1 \cap N_2)$ is Noetherian / Artinian. By the second isomorphism theorem $N_1/(N_1 \cap N_2) \simeq (N_1 + N_2)/N_2 \subseteq M/N_2$ is a submodule of a Noetherian / Artinian module, hence its Noetherian, as we wanted.

2.9 Problem 9

Let M be an A-module, $f : M \to M$ an A-endomorphism. Prove:

1. **If M is Noetherian and f surjective then f is an isomorphism.**
2. **If M is Artinian and f injective then f is an isomorphism.**

Part (1). Consider the ascending chain of submodules $\text{Ker} f \subseteq \text{Ker} f^2 \subseteq \cdots$. Let n be its stabilizing point, $\forall m \geq n \;\; \text{Ker} f^n = \text{Ker} f^m$. At this point $\text{Ker} f^n \cap \text{Im} f^n = 0$. Suppose not, then $\exists k \neq 0$ such that $f^n(k) = x \neq 0$ and $f^n(x) = 0$, so $f^{2n}(k) = 0$, contradicting that $\text{Ker} f^n = \text{Ker} f^{2n}$. Since f is epi, $\text{Im} f^n = M$, so $\text{Ker} f^n = 0$. Again using surjectivity of f^{n-1}, for all $m \in M$ there is a $m' \in m$ such that $f^{n-1}(m') = m$ so $f(m) = f^n(m') = 0$, hence $\text{Ker} f = 0$, so f is mono.

Part (2). The descending chain $\text{Im} f \supseteq \text{Im} f^2 \supseteq \cdots$ and using that M stabilizes at $\text{Im} f^n = \text{Im} f^{n+1}$. Hence, for all x, there is a k such that $f^{n+1}(k) = f^n(x)$. By injectivity $f(k) = x$, which implies f is epi.

2.10 Problem 10

Compute $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Z}), \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}), \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Q})$**.**

$\boxed{\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0}$ Suppose $f(1) = n \neq 0$. Then $(n+1)f(\frac{1}{n+1}) = f(1) = n$ so $n+1$ should divide n which is contradictory, $n + 1 > n$. Hence $f(1) = 0$, in which case $bf(a/b) = f(a) = 0f(a/b) = 0$, so $f \equiv 0$.

$\boxed{\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}}$. If $f(1) = q$ then $bf(a/b) = f(a) = f(1 + \cdots + 1) = af(1) = aq$, so $f(t) = qt$. Denote f_q the morphism $f_q(1) = q$. Then $f_{q+q'}(t) = (q + q')(t) = f_q(t) + f_{q'}(t)$ so the operator of the abeliean groups coincide, which finishes the proof.

$\boxed{\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}) = 0}$. Again lets see $f(1) = q \neq 0$ is contradictory. This time because $f(m) = f(0) = 0 \neq mq$. Hence 0 is the only plausible morphism.

2.11 Problem 11

Let A be a ring, M an A-module and $I \subseteq A$ an ideal. Prove $M/IM \simeq A/I \otimes_A M$.

We start with the short sequence $0 \mapsto I \overset{f}{\mapsto} A \overset{g}{\mapsto} A/I \mapsto 0$. Because $- \otimes_A M$ is right exact, we get the exact sequence $I \otimes_A M \overset{f'}{\mapsto} A \otimes_A M \overset{g'}{\mapsto} A/I \otimes_A M \mapsto 0$. From the isomorphism theorem over g'

we get $A/I \otimes_A M \simeq (A \otimes_A M)/(\text{Ker}g') = (A \otimes_A M)/(\text{Im}f') = (A \otimes_A M)/(I \otimes_A M)$. It remains to be seen that $(A \otimes_A M)/(I \otimes_A M) \simeq M/IM$.
Let $\phi : A \otimes_A M \mapsto M/IM$ be the map $\phi(a \otimes m) = \overline{am}$. Clearly $\text{Ker}\phi \supseteq I \otimes_A M$ as $\phi(i \otimes m) = \overline{im} = 0$. Conversely, if $\overline{am} = 0 \implies am = im' \implies m \in IM$. Hence $a \otimes m = a \otimes (im'') = (ia) \otimes m'' \in I \otimes_A M$, concluding $\text{Ker}\phi = I \otimes_A M$. By the isomorphism theorem, $M/IM \simeq (A \otimes_A M)/(I \otimes_A M)$, as we wanted.

2.12 Problem 12

Let A be a ring and $I, J \subseteq A$ ideals. Prove $A/I \otimes_A A/J \simeq A/(I + J)$.

From Problem 11, we have $A/I \otimes_A A/J \simeq (A/J)/(I(A/J))$. Now, define $\phi : A/J \mapsto A/(I + J)$ such that $\phi(\tilde{a}) = \overline{a}$, which is well defined as if $a - b \in I \implies a - b \in I + J$. Clearly it is exaustive as for all $\overline{a}$ there is a $\tilde{a}$ such that $\phi(\tilde{a}) = \overline{a}$.

Let's prove that $\text{Ker}\phi = I(A/J)$ and, by the isomorphism theorem, we will have finished.

$$\tilde{a} \in \text{Ker}\phi \iff a \in I + J \iff a = i + j \iff \tilde{a} = \tilde{i} + \tilde{j} = \tilde{i} = i \cdot \tilde{1} \in I(A/J)$$

2.13 Problem 13

Let A be a ring, M, N finitely generated A-modules. Prove:

1. **$M \otimes_A N$ is a finitely generated A-module.**
2. **If A is Noetherian, then $\text{Hom}_A(M, N)$ is a finitely generated A-module.**

Part (1). Let $M = \langle m_i \mid i \in [n] \rangle$ and $N = \langle n_j \mid j \in [m] \rangle$. We will see that

$$M \otimes_A N = \langle m_i \otimes n_j \mid i, j \in [n] \times [m] \rangle$$

For any $m \in M$, $n \in N$ $m \otimes n = (\sum a_i m_i) \otimes (\sum b_j n_j) = \sum \sum a_i b_j (m_i \otimes n_j)$.

Part (2). Because M if f.g, a morphism from M is uniquely defined by the images of the generators. Hence, there is a morphism inclusion $\text{Hom}(M, N) \mapsto N^n$. Because N is f.g, N^r is f.g too. As $\text{Hom}(M, N)$ is (isomorphic to) a submodule of a f.g module over a Noetherian ring, it is also finitely generated.

In the last step we have used the following proposition. If A is Noetherian and M is a finitely generated then M is Noetherian (hence any submodule, N, is finitely generated). This is just because a f.g module M is isomorphic to A^r/N which is Noetherian by the product and quotient closure properties.

2.14 Problem 14

Let A be a local ring, M, N finitely generated A-modules. Prove that $M \otimes_A N = 0$ if and only if $M = 0$ or $N = 0$. Prove that the result is no longer true if the ring is not local.

Construct the following morphism of A-modules $\phi : M \otimes_A N \mapsto M/mM \otimes_A N/mN$. Such that $\phi(m \otimes n) = \overline{m} \otimes \overline{n}$. By the properties of quotiens and tensor products, it is a morphism. Also, it is exhaustive as for any $\overline{m} \otimes \overline{n}$, there is a $m \otimes n$ that maps there.

Now if $M \otimes_A N = 0$, $\phi(0) = 0 \underset{\text{by exh.}}{=} M/mM \otimes_A N/mN \simeq M/mM \otimes_{A/m} N/mN$ which are vector spaces over the field A/m. This implies, WLOG $M/mM = 0 \implies M = mM$ which by Nakayama's Lemma implies $M = 0$.

The reverse implication is straight forward.

If the ring is not local this is not the case. If we have two maximal ideals I, J we can take the A-modules $A/I \neq 0$ and $A/J \neq 0$ but we will see that $A/I \otimes_A A/J = 0$. Take any $\tilde{a} \otimes \overline{a'}$ and any $i \in I \setminus J$. As A/J is a field, J is maximal, $\overline{i}$ has an inverse $\overline{r}$. Then $\tilde{a} \otimes \overline{a'} = \tilde{a} \otimes \overline{ira'} = \tilde{ia} \otimes \overline{ra'} = 0$ as $ia \in I$.

2.15 Problem 15

Let M be a finitely generated A-module and let $S \subseteq A$ be a multiplicatively closed set. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

$\boxed{\implies}$ If $S^{-1}M = 0$ Then for a fixed m, $m/1 = 0/1 \implies \exists t_m \in S \;\; t_m m = 0$. Now we can use f.g property and take the $s = \prod^n t_{m_i}$. With this s, $sm = \prod t_{mi}(\sum a_i m_i) = 0$.

$\boxed{\impliedby}$ If there exists one such s, for all element of $m/s' \in S^{-1}M$, $m/s' \sim 0/1$ as there exists a $t = s$ such that $s(m - s'0) = sm = 0$.

2.16 Problem 16

Let $S \subseteq A$ be a multiplicatively closed set. Prove that the localization functor S^{-1}- is exact.

We have seen in class that the functor $S^{-1}-$ is functor-isomorphic to $S^{-1}A \otimes_A -$ and that the functor $N \otimes_A -$ is right exact. So we only need to prove that if $g : M' \mapsto M$ is mono, then $S^{-1}(g) : S^{-1}M' \mapsto S^{-1}M$ is also mono, where $S^{-1}(g)(m/s) = g(m)/s$. If $S^{-1}(g(m'/s)) = 0 \implies g(m')/s = 0 \implies \exists t \in S$ such that $tg(m') \underset{t \in S \subseteq A}{=} g(tm') = 0 \implies tm' = 0 \implies m/s = 0$.

2.17 Problem 17

Let M be an A-module. We say that it is simple if it doesn't contain any non-trivial submodule (i.e. if $N \subseteq M$ is a submodule, then $N = 0$ or $N = M$). Prove:

1. **Every simple module is cyclic.**
2. **If M, N are simple A-modules and $f : M \to N$ is an homomomorphism, then $f = 0$ or f is an isomorphism.**

Part (1). For any $m \neq 0$, $\langle m \rangle$ is a non trivial submodule, so it must be the complete module. Hence M is ciclic.

Part (2). Let $f : M \mapsto N$ be a morphism of A-modules. Because $\text{Ker} f \subset M$ is a submodule, it must be either the complete of the zero. If it is the complete $\text{Ker} f = M \implies f \equiv 0$. If not, f is mono.

On the other hand, if $M = \langle m \rangle$ and $N = \langle n \rangle$, $\langle f(m) \rangle$ is a submodule of N, so it must either be the zero or the total. If it is the zero, $f(m) = 0$ so $f(m') = f(am) = af(m) = 0 \implies f \equiv 0$. If it is the total, for all $n' \in N$, $\exists a \in A$ such that $af(m) = n' \implies f(am) = n'$ so f is exhaustive.

As modules are an abelian category, mono and epi imply iso.

2.18 Problem 18

18. Let A be an integral domain and let M be an A-module. We say that $m \in M$ is a torsion element if there exists $a \in A$
$\{0\}$ such that $am = 0$. Let $T(M)$ be the set of torsion elements. Prove:

1. **$T(M)$ is a submodule of M**
2. **$M/T(M)$ has no torsion.**
3. **If $f : M \to N$ is A-linear, then $f(T(M)) \subseteq T(N)$.**
4. **If $0 \to M' \to M \to M'' \to 0$ is an exact sequence then $0 \to T(M') \to T(M) \to T(M'')$ is exact.**

Part (1). $m, m' \in T(M)$ then there are $a, a' \in A \setminus \{0\}$ such that $am = a'm' = 0$. Taking $aa' \neq 0$, because A is integral domain, and $aa'(m + m') = 0$, so $m + m' \in T(M)$.

Part (2). $\overline{m} \in T(M/T(M)) \implies \exists a \in A \setminus \{0\}$ such that $a\overline{m} = \overline{0} \implies am \in T(M)$. This implies $\exists b \in A \setminus \{0\}$ such that $bam = 0 = (ba)m$. $ba \neq 0$ because A is integral, so $m \in T(M) \implies \overline{m} = \overline{0}$.

Part (3). $x \in f(T(M))$ means $\exists m \in T(M)$ such that $f(m) = x$ and $\exists a \in A \setminus \{0\}$ such that $am = 0$. Hence $f(m) = x \implies ax = af(m) = f(am) = f(0) = 0$, so $x \in T(N)$.

Part (4). Start with an exact short sequence $0 \mapsto M' \overset{f}{\mapsto} M \overset{g}{\mapsto} M'' \mapsto 0$. Applying the functor T, we obtain $0 \mapsto T(M') \overset{T(f)}{\mapsto} T(M) \overset{T(g)}{\mapsto} T(M'')$ where $T(f) : T(M') \mapsto T(M)$ such that $T(f)(m') = f(m')$, which is well defined as $f(T(M')) \subseteq T(M)$. To see the the functor is left-exact, we must prove

1. f mono $\implies T(f)$ mono. $T(f)(m'_1) = T(f)(m'_2) \implies f(m'_1) = f(m'_2) \underset{\text{by inj.}}{\implies} m'_1 = m'_2$
2. $\text{Im} f = \text{Ker} g \implies \text{Im} T(f) = \text{Ker} T(g)$.

$$m \in \text{Ker} T(g) \iff T(g)(m) = 0 \iff g(m) = 0 \iff m \in \text{Ker} g \underset{\text{by hyp.}}{\iff} m \in \text{Im} f \iff$$

$$\iff \exists m' \;\; f(m') = m \iff T(f)(m') = m \iff m \in \text{Im} T(f)$$