Artin's conjecture on primes with prescribed primitive roots

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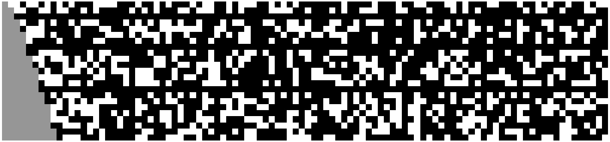
April 29, 2023

Objective

Conjecture (Artin's Conjecture)

Given $a \in \mathbb{Z}$, $a \notin \{-1, 0, 1\} \cup \{k^2 \mid k \in \mathbb{Z}_{>1}\})$, there are infinitely many primes p such that a is a primitive root in $(\mathbb{Z}/p\mathbb{Z})^*$.



- Rows: $a \in \mathbb{Z}$, increasing from top to bottom.
- Columns: primes p, increasing from left to right.
- A cell is white if a is a primitive root $\mod p$

History

1927. Emil Artin proposes a precise density conjecture.

1937. Herbert Bilharz solves the equivalent problem for $\mathbb{F}_q[x]$.

1957. Emma and Derrick H. Lehmer observe that the conjectured density is incorrect.

1967. Christopher Hooley proves AC under GRH.

1983. Rajiv Gupta and Ram Murty give a set of 13 integers such that at least one of them follows AC.

1985. Heath-Brown improves their argument to $\{2, 3, 5\}$.

History

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Part 1

Artin's Observation

A precise conjectured density

Artin's Observation

Conjecture (Artin's conjectured density)

Given a non-square integer $a \in \mathbb{Z}_{>1} \setminus \mathbb{Z}^2$, the density of primes where a is a primitive root is

$$A(a) = \delta(a) \prod_{l \text{ prime}} \left(1 - \frac{1}{l(l-1)}\right) \approx 0.3739558\ldots \cdot \delta(a)$$

where $\delta(a)$ is an explicit correction factor that is 1 for most a.

Without loosing much flavor, we may assume $a = 2$, which makes $\delta(a) = 1$.

Key Lemma

Observation

a is a primitive root mod p if and only if there isn't any $l \in \mathbb{Z}$ prime such that

$$\text{(1) } l \mid p-1 \qquad \text{and} \qquad \text{(2) } a^{\frac{p-1}{l}} = 1 \mod p$$

- Given (1), (2) $\iff x^l = a \mod p$ has a solution

Lemma (Key Lemma)

A prime l follows the conditions (1) and (2) for $p > 2$ if and only if p is completely split over $\mathbb{Q}(\zeta_l, a^{1/l}) = \mathit{SplField}_{\mathbb{Q}}(x^l - a)$.

- By Chebotarev's Density Theorem, they have density $\frac{1}{[\mathbb{Q}(\zeta_l, a^{1/l}):\mathbb{Q}]}$

Inclusion-Exclusion

Let k be square-free positive integer.

Theorem (Artin's observation)

The density of primes for which there is no $l \mid k$ following the conditions (1) and (2) is

$$A_k(a) = \sum_{d|k} \frac{\mu(d)}{[\mathbb{Q}(\zeta_d, a^{1/d}) : \mathbb{Q}]}$$

where μ is the Möebius Inversion function.

- Conjecture:

$$\lim_{k \text{ primordial}} A_k(a) = A(a)$$

- This *passing to limit* is where the difficulty lies.

Computation of the degree

Lemma

For $a = 2$, $[\mathbb{Q}(\zeta_k, a^{1/k}) : \mathbb{Q}] = \varphi(k)k$.

This is where Artin's original statement was incorrect for some values of a.

$$
\begin{tikzcd}[arrows=dash]
 & & L_k = \mathbb{Q}(\zeta_k, \sqrt[k]{a}) \arrow[dll] \arrow[drr] \arrow[dd] & & \\
C_k = \mathbb{Q}(\zeta_k) \arrow[drr] \arrow[ddrr, "\varphi(k)"'] & & & & R_k = \mathbb{Q}(\sqrt[k]{a}) \arrow[dll] \arrow[ddll, "k"] \\
 & & I_k = \mathbb{Q}(\zeta_k) \cap \mathbb{Q}(\sqrt[k]{a}) \arrow[d] & & \\
 & & \mathbb{Q} & &
\end{tikzcd}
$$

Summing up: Artin's Observation

- Encode l being a witness as a splitting condition over $\mathbb{Q}(\zeta_l, a^{1/l})$
- Chebotarev's Density Theorem
- Inclusion-Exclusion
- Conjectured *passing to the limit*

Part 2

Hooley's Theorem

The Riemann Hypothesis solves the problem

Hooley's Theorem

Theorem (Hooley, 1967)

The Generalized Riemann Hypothesis over the Number Fields $\mathbb{Q}(\zeta_k, a^{1/k})$ *imply Artin's Conjecture about the density of primes with a prescribed primitive root at* $a \in \mathbb{Z}_{>1} \setminus \mathbb{Z}^2$.

Sketch of the proof

- Sieve primes by intervals
- Reduce to the problem of counting prime ideals with bounded norm
- Result on vertical distribution of Riemann Zeroes under GRH

Hooley's Sieve

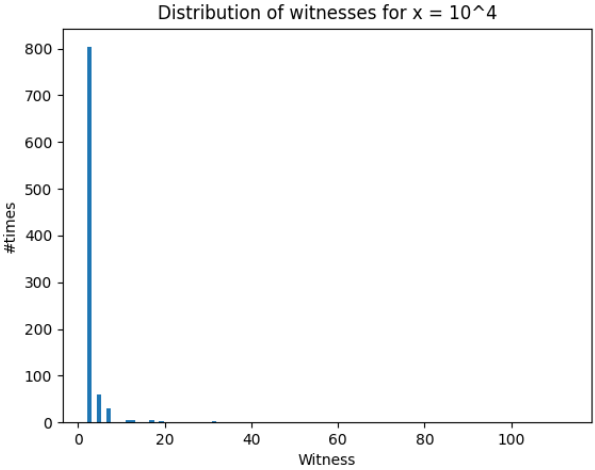
Definition (Prime counting functions)

1. $N_a(x) = \#\{p < x \mid a \text{ is a p.r. } \mod p\}$
2. $N_a(x, \xi) = \#\{p < x \mid \nexists\, q \text{ following (1 \& 2) in the range } q < \xi\}$
3. $M_a(x, \xi_1, \xi_2) =$
$$= \{p < x \mid \exists\, q \text{ following (1 \& 2) in the range } \xi_1 < q \leq \xi_2\}$$

Lemma

Let $\xi_1 = \frac{1}{6} \log x, \xi_2 = x^{1/2} \log^{-2} x, \xi_3 = x^{1/2} \log x$, *then*

$$N_a(x) = \underbrace{N_a(x, \xi_1)}_{\sim A(a) \frac{x}{\log x}} + \underbrace{O(M_a(x, \xi_1, \xi_2))}_{\preccurlyeq \frac{x}{(\log x)^2}} + \underbrace{O(M_a(x, \xi_2, \xi_3))}_{\preccurlyeq \frac{x \log \log x}{(\log x)^2}} + \underbrace{O(M_a(x, \xi_3, x - 1))}_{\preccurlyeq \frac{x}{(\log x)^2}}$$



For most p where a is not a primitive root, a is an l-th residue for l small.

Reduction to counting primes

Definition (Prime counting function)

For $k \in \mathbb{Z}_{>0}^{\text{square-free}}$, let $L_k = \mathbb{Q}(\sqrt[k]{a}, \zeta_k)$ and $n(k) = [L_k : \mathbb{Q}]$. Then, define

$$\pi(x,k) := \#\{\mathfrak{p} \text{ prime ideal of } L_k \mid \mathcal{N}\mathfrak{p} \leq x\}$$

Almost all prime ideals come from totally split primes in $\mathbb{Q}$

Lemma

Then,

$$n(k)P_a(x,k) \leq \pi(x,k) \leq n(k)P_a(x,k) + \underbrace{n(k)w(k)}_{e_p>1} + \underbrace{n(k)x^{1/2}}_{f_p>1}$$

where $w(k)$ is the number of unique prime factors of k.

- L_k is Galois $\implies p\mathcal{O}_{L_k} = \mathfrak{p}_1^{e_p} \dots \mathfrak{p}_{g_p}^{e_p}$ with $f_p = [\mathcal{O}_{L_k}/\mathfrak{p}_i : \mathbb{F}_p]$
- $\mathcal{N}(\mathfrak{p}_i) = p^{f_p} \leq x \implies p \leq x^{1/f_p}$

Effective prime counting estimate

Theorem (Main Theorem)

Assuming the Generalized Riemann Hypothesis for $\zeta_{L_k}(z)$, we have the estimate

$$\pi(x,k) = \frac{x}{\log x} + O(n(k)x^{1/2}\log(kx))$$

- $\pi(x,k)$ can be computed from the Riemann Zeroes
- Result about the vertical distribution of Riemann Zeroes under GRH.

Summing up: Hooley's Theorem

Key ideas

- For most p where a is not a primitive root, a is an l-th residue for l small $\implies$ Sieve
- Primes that come from unramified rational primes are dense $\implies$ Prime counting

Final Cannon

- Prime counting under GRH

Thank you for your attention

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Annex

Extra Slides

Motivation. *Disquisitiones Arithmeticae* 314-317

Why does the decimal expression of $\frac{3}{7}$ have a period of length 6, while the expression of $\frac{1}{11}$ has a shorter period, of only 2 digits?

$$\frac{3}{7} = 0.428571\ 428571\ 428571\ldots \qquad \frac{1}{11} = 0.09\ 09\ 09\ldots$$

Remark

For p a prime and $a \in \mathbb{Z} \cap [1, p-1]$, the length of the decimal period of $\frac{a}{p}$ is $\mathrm{ord}_{(\mathbb{Z}/p\mathbb{Z})^\times}(10)$.

$$\frac{a}{p} = \left(\frac{a_1}{10} + \cdots + \frac{a_s}{10^s}\right)\left(1 + \frac{1}{10^s} + \cdots\right) = \left(10^{s-1}a_1 + \cdots + a_s\right)\frac{1}{10^s - 1}$$

$$a(10^s - 1) = Mp \implies 10^s = 1 \mod p$$

Motivation II. *Disquisitiones Arithmeticae* 314-317

Remark

Given $a, b \in \mathbb{Z} \cap [1, p-1]$ such that $b = 10^\lambda a \mod p$ for some λ, then period of $\frac{b}{p}$ is a cyclic translation of the period of $\frac{a}{p}$.

$$b_i = \left\lfloor \frac{10^i b}{p} \right\rfloor \mod 10 = \left\lfloor \frac{10^i (10^\lambda a + Np)}{p} \right\rfloor \mod 10 = a_{i+\lambda}$$

Question

For which primes p are the periods of $\frac{a}{p}$ all translations of the period of $\frac{1}{p}$?

This is tantamount to asking for which primes is 10 a primitive root.

Gauss' Party Trick

$\frac{1}{7}$	0.142857	142857	$\ldots$
$\frac{2}{7}$	0.2857	142857	$\ldots$
$\frac{3}{7}$	0.42857	142857	$\ldots$
$\frac{4}{7}$	0.57	142857	$\ldots$
$\frac{5}{7}$	0.7	142857	$\ldots$
$\frac{6}{7}$	0.857	142857	$\ldots$

Background I. Number Fields

A Number Field K is a finite field extension of $\mathbb{Q}$.

Given a Number Field, one can define its ring of integers $\mathcal{O}_K$, which is a generalization of $\mathbb{Z} \subseteq \mathbb{Q}$.

$$\begin{tikzcd} K \arrow[d, dash] & \mathcal{O}_K \arrow[l, hook'] \arrow[d, dash] \\ \mathbb{Q} & \mathbb{Z} \arrow[l, hook'] \end{tikzcd} \qquad \begin{tikzcd} p\mathcal{O}_K = \mathfrak{p}_1 \dots \mathfrak{p}_n \arrow[d, dash] \\ p \end{tikzcd}$$

In these rings, factorization of ideals as a product of prime ideals is unique.

Definition (Completely split prime)

A prime p is called completely split over K if $\mathfrak{p}_i \neq \mathfrak{p}_j$ for $i \neq j$ and the residue fields $(\mathcal{O}_K/\mathfrak{p}_i) \simeq \mathbb{F}_p$

Background. Dirichlet's Density

Inspired by Dirichlet's theorem about primes in arithmetic progressions.

$$\sum_{p=an+b \text{ prime}} \frac{1}{p}$$

Definition (Dirichlet's Density)

For $S \subseteq \operatorname{Spec} \mathcal{O}_K$, define

$$\delta(S) = \lim_{s \to 1} \frac{\sum_{\mathfrak{p} \in S} \frac{1}{(\mathcal{N}\mathfrak{p})^s}}{\sum_{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K} \frac{1}{(\mathcal{N}\mathfrak{p})^s}} \tag{1}$$

Good number theoretical density because it can often be related with special values of L-functions.

Background III. Chebotarev's Theorem

Theorem (Chebotarev's Density Theorem. Simplified Version)

Let $K/\mathbb{Q}$ be a finite Galois extension. The Dirichlet Density of the set S of primes $\mathfrak{p} \subseteq \mathbb{Q}$ that are totally split over K is

$$\delta(S) = \frac{1}{[K : \mathbb{Q}]}$$

For example, when $K = \mathbb{Q}(\zeta_n)$ a prime splits completely if and only if $p = 1 \mod n$. They have density $\frac{1}{\varphi(n)}$.

Artin's Observation III. Chebotarev's theorem

Let k be square-free positive integer.

Lemma

All the primes $l \mid k$ follow the conditions (1) and (2) for $p > 2$ if and only if p is completely split over $L_k/\mathbb{Q}$, where $L_k = \prod_{l|k} L_l = \mathbb{Q}(\zeta_k, a^{1/k})$.

Chebotarev's theorem yields that the density of

$$\{p \mid p > 2 \text{ prime such that } \forall l \mid k \text{ conditions (1) and (2) are met}\}$$

is $\frac{1}{[L_k:\mathbb{Q}]}$.

Hooley's Theorem II. Prime counting functions

Definition (Prime counting functions)

1. $N_a(x) = \#\{p < x \mid a \text{ is a p.r. } \mod p\}$
2. $P_a(x, k) = \#\{p < x \mid \forall q \mid k,\ q \text{ follows } (1 \ \& \ 2)\}$
3. $N_a(x, \xi) = \#\{p < x \mid \ \nexists\, q \text{ following } (1 \ \& \ 2) \text{ in the range } q < \xi\}$
4. $M_a(x, \xi_1, \xi_2) =$
   $$= \{p < x \mid \exists\, q \text{ following } (1 \ \& \ 2) \text{ in the range } \xi_1 < q \leq \xi\}$$

Lemma (Artin's Observation)

$$N_a(x, \xi) = \sum_{l'} \mu(l') P_a(x, l')$$

as l' goes over square-free integers with all prime factors $\leq \xi$.

Estimation of term 1

Lemma (Estimation of the 1st term)

$$N_a(x,\xi_1) = \sum_{l'} \mu(l')\left(\frac{x}{\log x \cdot n(l')} + O(x^f \log x)\right) =$$

$$\underset{l'<e^{2\xi_1}}{=} \frac{x}{\log x}\sum_{l'}\frac{\mu(l')}{n(l')} + O\left(\sum_{l<e^{2\xi_1}} x^f \log x\right) =$$

$$= A(a)\frac{x}{\log x} + O(e^{2\xi_1}x^f \log x) =$$

$$= A(a)\frac{x}{\log x} + O(x^{f+1/3}\log x)$$

Bound of term 2

Lemma (Bound of the 2nd term)

$$M_a(x,\xi_2,\xi_3) \le \sum_{\xi_1 < q \le \xi_2} \left(\frac{x}{\log x \cdot q(q-1)} + O(x^f \log x) \right) =$$

$$= O\left(\frac{x}{\log x} \sum_{q > \xi_2} \frac{1}{q^2} \right) + O\left(x^f \log x \sum_{q \le \xi_2} 1 \right) =$$

$$= O\left(\frac{x}{\xi_1 \log x} \right) + O\left(\frac{x^f \xi_2 \log x}{\log \xi_2} \right) = O\left(\frac{x}{\log^2 x} \right)$$

Bound of term 3

Lemma (Bound of the 3rd term)

Let $\xi_2 = x^{1/2}\log^{-2} x$ *and* $\xi_3 = x^{1/2}\log x$. *Then* $M_a(x,\xi_2,\xi_3) = O\left(\frac{x}{\log^2 x}\right)$.

In particular $p \equiv 1 \mod q$. By Brun's method, which is an inequality related to Dirichlet's Theorem, we have

$$P_a(x,q) \leq \sum_{\substack{p\leq x \\ p\equiv 1 \mod q}} 1 \leq \frac{A_1 x}{(q-1)\log(x/q)}$$

$$\begin{aligned} M_a(x,\xi_2,\xi_3) &= O\left(\frac{x}{\log x}\sum_{\xi_2<q\leq\xi_3}\frac{1}{q}\right) = \\ &= O\left(\frac{x}{\log^2 x}\left(\log\frac{\xi_3}{\xi_2}+O(1)\right)\right) = O\left(\frac{x\log\log x}{\log^2 x}\right) \end{aligned}$$

Bound of term 4

Lemma (Bound of the 4th term)

Let $\xi_3 = x^{1/2}\log x$, *then*

$$M_a(x, \xi_3, x-1) = O\left(\frac{x}{\log^2 x}\right) \tag{2}$$

In particular $a^{\frac{p-1}{q}} = 1 \mod p$. Hence, if there is a $q > \xi_3$ that follows the Lemma, there will be an $m < \frac{x}{\xi_3}$ such that $p | a^m - 1$. All the primes counted on $M_a(x, \xi_3, x-1)$ need to be divisors of

$$S_a(x/\xi_3) := \prod_{m < x/\xi_3} (a^m - 1)$$